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Pairwise Compatible Hamilton Decompositions of K_n

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In this paper we prove that there are either $2k-2$ or $2k-1$ pairwise compatible Hamilton path decompositions of K_{2k}. In the case of K_4, there exactly 2 compatible Hamilton path decompositions. We also find (different) lower bounds on the

1. INTRODUCTION AND NOTATION

Definitions not included here are as in [3, 8]. Let G be a finite graph. A 2-*path* is a sequence v_0, v_1, v_2, where $v_0, v_1, v_2 \in V(G)$, and $v_0 \neq v_1 \neq v_2 \neq v_0$. We write it as $v_0 v_1 v_2$ and say it is *centred at* v_1 with *end vertices* v_0 and v_2. The proofs in this paper rely on the fact that a trail or a tour can be described uniquely by the set of 2-paths it contains. We call two trails (or tours) t_1 and t_2 in G *similar* if there exists an automorphism ρ of $V(G)$ such that for all $v_0, v_1, v_2 \in V(G)$, $v_0 v_1 v_2$ is a 2-path in t_1 if and only if $\rho(v_0 v_1 v_2) = \rho(v_0)\,\rho(v_1)\,\rho(v_2)$ is a 2-path in t_2. We call two Hamilton decompositions of K_{2k+1} *compatible* if they have no 2-path in common. Similarly, we call two Hamilton path decompositions of K_{2k} *compatible* if they have no 2-path in common. We call a set of $2k-1$ pairwise compatible Hamilton (path) decompositions of $K_{2k+1}(K_{2k})$, a *perfect set of Hamilton* (*path*) *decompositions* of $K_{2k+1}(K_{2k})$.

The results in this paper were motivated by a question of Kotzig's [6]:

Problem 1 (Kotzig [6]). What is the smallest $k > 1$ for which there is a perfect set of Hamilton decompositions of K_{2k+1}?

It is possible that no such k exists. It is not hard to show that there cannot be two compatible Hamilton decompositions of K_5, let alone three

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pairwise compatible ones, where three is the number needed for a perfect set. Kotzig states in [6] that it is known that K_7 does not have a perfect set of Hamilton decompositions, but does not say how many pairwise compatible Hamilton decompositions are possible. The fact that perfect sets of Hamilton decompositions do not exist for these small cases leads us to ask instead:

PROBLEM 2. Given k, what is the maximum number of pairwise compatible Hamilton decompositions in K_{2k+1}?

Since a set of l pairwise compatible Hamilton decompositions of K_{2k+1} implies the existence of a set of l pairwise compatible Hamilton path decompositions of K_{2k}, we can back up still further and ask:

PROBLEM 3. Given k, what is the maximum number of pairwise compatible Hamilton path decompositions in K_{2k}?

It is easy to show that K_4 has two compatible Hamilton path decompositions but does not have a perfect set of Hamilton path decompositions. Nothing else was known for larger k.

A Dudeney set of K_n is a set of Hamilton cycles of K_n that between them contain every 2-path of the group exactly once. Problems 1 and 2 are related to the existence of Dudeney sets in K_{2k+1} because a perfect set of Hamilton decompositions of K_{2k+1} is simply a resolvable Dudeney set. Also, since whenever there exists a Dudeney set of K_n, we immediately have a set of Hamilton paths of K_{n-1} that partition the 2-paths of K_{n-1}, results about Dudeney sets may have implications for Problem 3. Since Dudeney sets in K_n when n is odd have proven hard to find, we should perhaps assume that solving Problem 1 will be difficult. There are only two known infinite families of Dudeney sets of K_{2k+1}:

THEOREM 1 (Nakamura, Kiyasu-Zen'iti, and Ikeno [7]). *There is a Dudeney set in K_n if $n=2^e+1$, where e is a natural number.*

THEOREM 2 (Heinrich, Kobayashi, and Nakamura [2]). *There is a Dudeney set on K_{p+2} if p is prime and 2 is a generator of the multiplicative subgroup of $GF(p)$.*

There are also a few sporadic cases known; see [4].

However, when n is even, the existence of Dudeney sets has been solved completely.

THEOREM 3 (Kobayashi, Kiyasu-Zen'iti, and Nakamura [4]). *There exists a Dudeney set in K_{2k}.*

Before proving Theorem 3, Kobayashi and Nakamura [5] gave an elegant construction of the following result.

THEOREM 4 (Kobayashi and Nakamura [5]). *There exists a set of Hamilton cycles of K_{2k} that between them contain every 2-path of K_{2k} exactly twice.*

As a corollary, there is a set of Hamilton paths of K_{2k-1} that between them contain every 2-paths of K_{2k-1} exactly twice. One of the corollaries of the result in Section 2 is to show that there also exists a set of Hamilton paths of K_{2k} that between them contain every 2-path of K_{2k} exactly twice.

This main result in Section 2 arises from addressing Problem 3 and consists of showing that for any k there are at least $2k-2$ pairwise compatible Hamilton path decompositions of K_{2k}. A simple corollary of the proof of this theorem is that there exists a set of $4k-2$ Hamilton path decompositions of K_{2k} such that every 2-path is in exactly two of the Hamilton paths. In Section 3 we obtain lower bounds on the number of pairwise compatible Hamilton decompositions of K_{2k+1}, one for the case k even and one for the case k odd.

We will describe a Hamilton decomposition of K_{2k+1} by providing a list of the 2-paths it contains, using the following notation to describe a set of 2-paths centred at a given vertex in K_{2k+1}. Let $V(K_{2k}) = \{\infty_1\} \cup \{0, 1, ..., 2k-2\}$ and $V(K_{2k+1}) = V(K_{2k}) \cup \{\infty\}$. Let F be a 1-factor of K_{2k}. By $\infty[F]$ we mean the set of k 2-paths of the form $a\infty b$, where $a\,b$ is an edge in F. Given $w \in V(K_{2k})$, by $w[F]$ we mean the set of $(k-1)$ 2-paths of the form $a\,w\,b$, where $a\,b$ is an edge in F and $a \neq w \neq b$, together with the single 2-path $\infty w\,b$, where $w\,b$ is an edge in F.

We will describe a Hamilton path decomposition of K_{2k} using the exact same notation, but with a slightly different interpretation. Obviously, in this case, we do not want the 2-paths that contain the vertex ∞. So, when we are considering 2-paths in K_{2k}, given $w \in V(K_{2k})$, we take $w[F]$ to mean only the set of $(k-1)$ 2-paths of the form $a\,w\,b$, where $a\,b$ is an edge in F and $a \neq w \neq b$. This use of one notation for two different meanings is not confusing as it will always be clear which graph we are considering. Also, it is common as it will always be clear which graph we are considering. Also, it is common practise to construct a Hamilton path decomposition of K_{2k} by removing a vertex (in our case, ∞) from a Hamilton decomposition of K_{2k+1}. That is all we are doing here.

We assume all addition is modulo $2k-1$ with residue classes $0, 1, ..., 2k-2$, unless otherwise stated. We now define a particular 1-factorization $\mathscr{F}$ of K_{2k}; let $\mathscr{F} = \{F_0, F_1, ..., F_{2k-2}\}$, where, for $0 \leqslant i \leqslant 2k-2$, F_i is the 1-factor $\{\infty_1 i\} \cup \{x\,y : x, y \in \{0, 1, ..., 2k-2\}, x \neq y$ and $x + y \equiv 2i$

(mod $2k-1$)}. Let σ be the permutation $(\infty_1)(0\,1\,2 \cdots 2k-2)$ of $V(K_{2k})$. Then $F_i = \sigma^i(F_0)$ for all $i \in \{0, 1, 2, ..., 2k-2\}$.

We define a "length" function on the edges in K_{2k} that do not contain vertex ∞_1 as follows. Let

$$\ell(x\, y) = \min((x-y)(\mathrm{mod}\ 2k-1), (y-x)(\mathrm{mod}\ 2k-1))$$

be the *length* of the edge $x\,y$. We say two edges v_1v_2 and u_1u_2 in K_{2k} are *parallel* if none of the vertices is ∞_1 and $u_1+u_2 \equiv v_1+v_2$ (mod $2k-1$). For example, for each $i \in \{0, 1, 2, ..., 2k-2\}$, the edges in F_i that do not contain ∞_1 are pairwise parallel.

Finally, we will use the notation (a, b) for the greatest common factor of two integers a and b, and $\phi(n)$ for the Euler ϕ function. We will use $2^{-1}a$ (mod $2k-1$) to indicate either $(a/2)$ (mod $2k-1$), if a is even, or $(a+2k-1)/2$ (mod $2k-1$), if a is odd. This is multiplication by 2^{-1} in the ring Z_{2k-1}.

2. PAIRWISE COMPATIBLE HAMILTON PATH DECOMPOSITIONS OF K_{2k}

The graph K_{2k} has $k(2k-1)(2k-2)$ 2-paths. A Hamilton path decomposition of K_{2k} contains $k(2k-2)$ 2-paths. We would like to construct a set of $2k-1$ pairwise compatible Hamilton path decompositions of K_{2k}: a perfect set of Hamilton path decompositions of K_{2k}. However, when $k=2$, it is possible to find at most two compatible Hamilton path decompositions. In Theorem 5 we extend this result by constructing $2k-2$ pairwise compatible Hamilton path decompositions of K_{2k} for all values of k. There is however no reason to suppose for $k>2$ that it is not possible to find $2k-1$ pairwise compatible Hamilton path decompositions.

THEOREM 5. *The complete graph K_{2k} has a set of $2k-2$ pairwise compatible Hamilton path decompositions for all $k>1$.*

We first prove two lemmas.

Suppose for some $a, b \in \{0, 1, ..., 2k-2\}$ that $F_a \cup F_b$ is a Hamilton cycle H of K_{2k}. We can assume that $H = (w_1 w_2 \cdots w_{2k})$, that the edge w_1w_2 is in F_a, and that $w_1 = \infty_1$. We want to consider the 2-paths in $\{w_{2j-1}[F_a] \cup w_{2j}[F_b] : 1 \leqslant j \leqslant k\}$, and we want to think of them as 2-paths in K_{2k}, (so we will disregard the 2-paths containing ∞ that would occur if we were considering the graph K_{2k+1}).

LEMMA 6. *Given that $F_a \cup F_b$ is a Hamilton cycle $H = (w_1 w_2 \cdots w_{2k})$ of K_{2k}, where $w_1 = \infty_1$ and $w_1w_2 \in F_a$, the trails in K_{2k} formed by the set of*

2-paths in $\{w_{2j-1}[F_a] \cup w_{2j}[F_b] : 1 \leqslant j \leqslant k\}$ *form a Hamilton path decomposition of* K_{2k}. *Also, the trail that starts on vertex* ∞_1 *ends on vertex* $2^{-1}(a+b)$.

Proof. The outer cycle in Fig. 1 is the Hamilton cycle $H = F_a \cup F_b$ when k is even. When k is odd, a similar figure is obtained.

The subtrail of $\{w_{2j-1}[F_a] \cup w_{2j}[F_b] : 1 \leqslant j \leqslant k\}$ in K_{2k} that starts on w_1 is the Hamilton path P given by the boldface edges. It is not hard to see that the trails that start on the other vertices form Hamilton paths in exactly the same way. In fact, if we let ρ be the following permutation of $V(K_{2k})$,

$$\rho = (w_1 w_2 \cdots w_{2k}),$$

then the other trails formed by the set of 2-paths in $\{w_{2j-1}[F_a] \cup w_{2j}[F_b] : 1 \leqslant j \leqslant k\}$ are $\rho^j(P)$, for $1 \leqslant j \leqslant k-1$.

By the definitions of F_a and F_b, we can describe vertices w_i, $2 \leqslant i \leqslant 2k$, in terms of a and b. The Hamilton path P shown in this figure starts at $w_1 = \infty_1$ and ends at $w_{k+1} \equiv kb - (k-1)\,a \equiv ka - (k-1)\,b \equiv 2^{-1}(a+b) \pmod{2k-1}$. ▮

The proof of the next lemma is heavily based on the proof of Theorem 1 in [1].

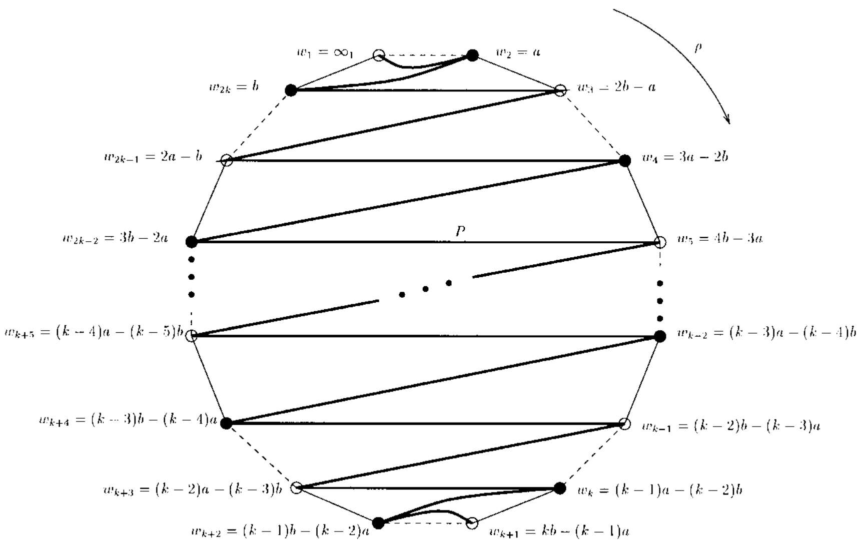

FIG. 1. P and ρ.

LEMMA 7. *Assume that* $c > d$, *where* $c, d \in \{0, 1, 2, ..., 2k-2\}$. *If* $c-d$ *and* $2k-1$ *are relatively prime, then* $F_c \cup F_d$ *is a Hamilton cycle, where* $F_i = \{\infty_1\, i\} \cup \{x\, y : x \neq y$ *and* $x + y \equiv 2i \pmod{2k-1}\}$, *for* $i \in \{c, d\}$.

Proof. Let F_c and F_d be two such 1-factors of K_{2k} such that $c-d$ and $2k-1$ are relatively prime. Consider an l-subset of those edges in F_c that do not contain ∞_1. The sum of the vertices in these edges will be congruent to $2lc \pmod{2k-1}$, since an edge xy in F_c, $x \neq \infty_1 \neq y$, satisfies $x + y \equiv 2c \pmod{2k-1}$. Similarly for F_d. Suppose $F_c \cup F_d$ is not a Hamilton cycle of K_{2k}. Then there is an even length $2m$-cycle in $F_c \cup F_d$ does not contain ∞_1, where $2 \leqslant m \leqslant k-1$. We can sum the vertices in this cycle as edges of F_c or as edges of F_d to get that $2mc \equiv 2md \pmod{2k-1}$. This contradicts the fact that $c-d$ and $2k-1$ are relatively prime. ▮

Now consider the particular 1-factors F_0 and F_1. Consider the following permutations of $V(K_{2k})$:

$$\sigma = (\infty_1)(0\ 1\ 2 \cdots 2k-2),$$

$$\tau = (\infty_1)(k)(0\ 1)(2\ 2k-2)(3\ 2k-3) \cdots (k-1\ k+1).$$

Note that $\tau(F_0) = F_1$ and $\tau(F_1) = F_0$.

We now prove Theorem 5 in a series of claims. Each of $H_0, H_1, ..., H_{k-2}$ and $H'_0, H'_1, ..., H'_{k-2}$ is going to be defined as a certain set of 2-paths, and our objective is to show that each of these sets of 2-paths is a Hamilton path decomposition of K_{2k}, and also that these Hamilton path decompositions are all pairwise compatible. To obtain this objective, we first list the 2-paths in H_0, show how to determine the H_j and H'_j so they are similar to H_0, then show that no two of $\{H_0, H_1, ..., H_{k-2}\} \cup \{H'_0, H'_1, ..., H'_{k-2}\}$ have a 2-path in common, and finally prove that H_0 is a Hamilton path decomposition of K_{2k}.

Define the 2-paths in H_0 to be

$$\infty_1[F_0],$$

$$0[F_1],$$

$$2i[F_0], \quad \text{for} \quad i \in \{1, 2, ..., k-1\},$$

$$(2i-1)[F_1], \quad \text{for} \quad i \in \{1, 2, ..., k-1\}.$$

Let $H'_0 = \tau(H_0)$, $H_j = \sigma^{2j}(H_0)$, for $1 \leqslant j \leqslant k-2$, and $H'_j = \sigma^{2j}(H'_0)$, for $1 \leqslant j \leqslant k-2$. By definition, the H_j and H'_j are all similar to H_0, so it enough to prove that H_0 is a Hamilton path decomposition to prove that they all are.

CLAIM 8. *The 2-paths in H'_0 are $\infty_1[F_1]$, $0[F_0]$, and $2i[F_1]$ and $(2i-1)[F_0]$ for $i \in \{1, 2, ..., k-1\}$.*

Proof. This follows immediately since $\tau(F_0) = F_1$ and $\tau(F_1) = F_0$. ∎

CLAIM 9. *For any $j \in \{0, 1, ..., k-2\}$, the set of 2-paths in H_j and H'_j contains every 2-path in K_{2k} with end vertices from an edge in F_{2j} or F_{2j+1} exactly once.*

Proof. By definition and by Claim 8, we know that H_0 and H'_0 between them contain every 2-path with end vertices from F_0 or F_1, exactly once. Let $j \in \{0, 1, ..., k-2\}$. Since $H_j = \sigma^{2j}(H_0)$ and $H'_j = \sigma^{2j}(H'_0)$, and $F_{2j} = \sigma^{2j}(F_0)$ and $F_{2j+1} = \sigma^{2j}(F_1)$, we know that H_j and H'_j between them contain every 2-path in K_{2k} with end vertices from an edge in F_{2j} or F_{2j+1} exactly once. ∎

It follows that no two of $\{H_0, H_1, ..., H_{k-2}\} \cup \{H'_0, H'_1, ..., H'_{k-2}\}$ have a 2-path in common. In fact we have all possible 2-paths exactly once except those with end vertices from an edge in F_{2k-2}.

CLAIM 10. *The 2-paths in H_0 form a Hamilton path decomposition of H_{2k}.*

Proof. By Lemma 7, $F_0 \cup F_1$ is a Hamilton cycle of K_{2k}. We can therefore use Lemma 6 to prove that the 2-paths in H_0 form a Hamilton path decomposition. ∎

This completes the proof of Theorem 5. ∎

It would seem to be difficult to find a perfect set of Hamilton path decompositions of K_{2k}. However, we can find a set of Hamilton path decomposition of K_{2k} that contain every 2-path exactly twice as a simple corollary to the proof of Theorem 5.

COROLLARY 11. *The complete graph K_{2k} has a set of $4k-2$ Hamilton path decompositions so that every 2-path in K_{2k} is in exactly two of them.*

Proof. Let $H_0, H_1, ..., H_{2k-2}$ and $H'_0, H'_1, ..., H'_{2k-2}$ be the Hamilton path decompositions we want to construct. Define H_0 and H'_0 as in the proof of Theorem 5. Let $H_j = \sigma^{2j}(H_0)$, $0 \leqslant j \leqslant 2k-2$, and $H'_j = \sigma^{2j}(H'_0)$, $0 \leqslant j \leqslant 2k-2$. Exactly as before, we can show that for all $j \in \{0, 1, ..., 2k-2\}$, H_j and H'_j between them contain every 2-path in K_{2k} with end vertices from an edge in F_{2j} or F_{2j+1}, where addition on the subscripts of the 1-factors is modulo $2k-1$, with residue classes $0, 1, ..., 2k-2$. ∎

It seems appropriate to mention the next two results as they tie in with the result in Theorem 4. The first is an obvious corollary of Corollary 11: the second is a corollary of Theorem 4 [5].

Corollary 12. *There exists a set of Hamilton paths of K_{2k} that between them contain every 2-path of K_{2k} exactly twice.*

Corollary 13 [5]. *There exists a set of Hamilton paths of K_{2k+1} that between them contain every 2-path of K_{2k+1} exactly twice.*

3. PAIRWISE COMPATIBLE HAMILTON CYCLE DECOMPOSITIONS

The graph K_{2k+1} has $k(2k+1)(2k-1)$ 2-paths. A Hamilton decomposition of K_{2k+1} contains $k(2k+1)$ 2-paths. We would like to construct a set of $2k-1$ pairwise compatible Hamilton decompositions of K_{2k+1}: a perfect set of Hamilton decompositions of K_{2k+1}.

In Section 2 we found a set of $2k-2$ pairwise compatible Hamilton path decompositions of K_{2k}. If the edges determined by the end vertices of each of the Hamilton paths were distinct, we could add a new vertex ∞ to each Hamilton path decomposition and join the ends of each Hamilton path with a 2-path centred at ∞ to construct $2k-2$ pairwise compatible Hamilton decompositions K_{2k+1}. Sadly this does not happen. In this section we again construct Hamilton path decompositions of K_{2k}, but this time we make sure that we after we add vertex ∞ to both ends of each Hamilton path, the 2-paths centred at ∞ needed to join each of these trails into a Hamilton cycle of K_{2k+1} will all be different. In Theorem 16 we construct $\lfloor (2k-1)/3 \rfloor$ pairwise compatible Hamilton decompositions of K_{2k+1} when $k>3$ is odd, and two compatible Hamilton decompositions of K_7. In Theorem 21 we construct $\max(\lceil 2k/3 \rceil - (k-1-\phi(2k-1)/2), 3)$ pairwise compatible Hamilton decompositions of K_{2k+1}, when $k>2$ is even.

If we now consider the 2-paths in Lemma 6 as 2-paths of K_{2k+1} (rather than K_{2k}), it is easy to see that the only difference is to add vertex ∞ to both ends of each of the Hamilton paths constructed in Lemma 6. The proofs of the next two lemmas are not given. The first follows easily from Fig. 1 and the proof of Lemma 6, and the second is trivial.

Lemma 14. *Again suppose that $F_a \cup F_b$ is the Hamilton cycle $H=(w_1 w_2 \cdots w_{2k})$ of K_{2k}, where $w_1=\infty_1$ and $w_1 w_2 \in F_a$. The trails in K_{2k+1} formed by the set of 2-paths in $\{w_{2j-1}[F_a] \cup w_{2j}[F_b]: 1 \leqslant j \leqslant k\}$ have the following properties*:

1. *Each trail starts and ends at vertex* ∞. *More precisely, each trail is simply one of the Hamilton paths constructed in Lemma* 6 *with vertex* ∞ *joined to each end.*

2. *The trail that begins on the edge* $\infty w_1 = \infty \infty_1$ *ends on the edge* $w_{k+1} \infty = 2^{-1}(a+b)\,\infty$.

3. *The trail that begins on the edge* ∞w_i, *for* $2 \leqslant i \leqslant k$, *ends on the edge* $w_{i+K} \infty$.

LEMMA 15. *Let* $u\,v$ *and* $x\,y$ *be two edges in* K_{2k} *such that none of the vertices is* ∞_1. *If* $u\,v$ *and* $x\,y$ *are not parallel, then* $2^{-1}(u+v) \not\equiv 2^{-1}(x+y)$ (mod $2k-1$).

Before considering the two cases of k odd and even separately, we first introduce one final piece of notation. Again assume that $F_a \cup F_b$ is the Hamilton cycle $H = (w_1 w_2 \cdots w_{2k})$ of K_{2k}, where $w_1 = \infty_1$ and $w_1 w_2 \in F_a$. By $E_{a,b}$ we mean the set of edges $\{w_i w_{i+k} : 1 \leqslant i \leqslant k\}$. Note that for $c, d \in V(K_{2k})$, the edge $c\,d$ is in $E_{a,b}$ if and only if one of the trails in K_{2k+1} formed by the 2-paths in $\{w_{2j-1}[F_a] \cup w_{2j}[F_b] : 1 \leqslant j \leqslant k\}$ starts on the edge $\infty\, c$ and ends on the edge $d\,\infty$. Note also that $E_{a,b} = E_{b,a}$.

We now assume k is odd and set $k = 2m+1$. We want to prove the following theorem.

THEOREM 16. *Let* $m > 1$. *There are at least* $\lfloor (4m+1)/3 \rfloor$ *pairwise compatible Hamilton decompositions of* K_{4m+3}. *There are at least two compatible Hamilton decompositions of* K_7.

By Lemma 7 we know that $F_a \cup F_{a+1}$ is a Hamilton cycle of K_{4m+2}. We can therefore use Lemma 14 with $b = a+1$. In order to join the two ends of each of the ensuing trails together to form a Hamilton cycle, we need to know which 2-paths we would use. In the following lemma, we investigate the edges in $E_{a,a+1}$.

LEMMA 17. *Let* $a \in V(K_{4m+2}) \backslash \{\infty_1\}$. *Then*

$$
\begin{aligned}
E_{a,a+1} = {} & \{\infty_1\,(2^{-1}(2a+1))\} \\
& \cup \{a\,(a-2m+1), (a-1)\,(a-2m+2), \ldots, (a-m+1)\,(a-m)\} \\
& \cup \{(a+1)\,(a+2m), (a+2)\,(a+2m-1), \ldots, (a+m)\,(a+m+1)\}.
\end{aligned}
$$

Proof. The edges listed in the statement of the lemma are shown in Fig. 2. We can, as usual, assume that $F_a \cup F_{a+1}$ is the Hamilton cycle $H = (w_1 w_2 \cdots w_{4m+2})$ of K_{4m+2}, where $w_1 = \infty_1$ and $w_1 w_2 \in F_a$. By Lemma 14 the edge $\infty_1(2^{-1}(2a+1))$ is in $E_{a,a+1}$. Also by Lemma 14, for each $i \in \{2, 3, \ldots, 2m+1\}$, the edge $w_i w_{i+2m+1}$ is in $E_{a,a+1}$. For $i \in \{2, 4, 6, \ldots, 2m\}$, $w_i \equiv a - i + 2 \pmod{4m+1}$. For each such i, $w_i + w_{i+2m+1} \equiv 2a + 2m + 2 \equiv 2a - 2m + 1 \pmod{4m+1}$. For $i \in \{3, 5, 7, \ldots, 2m+1\}$, $w_i \equiv a + i - 1 \pmod{4m+1}$. For each such i, $w_i + w_{i+2m+1} \equiv 2a - 2m \equiv 2a + 2m + 1 \pmod{4m+1}$. ▮

Finally, we want to maximize the number of pairs of 1-factors on which we can use Lemma 14.

LEMMA 18. *For $m > 1$, there is a set S of disjoint edges of length 1 in K_{4m+2} so that $|S| = \lfloor (4m+1)/3 \rfloor$ and so that if the edges $u(u+1)$ and $v(v+1)$ are in S, then $E_{u,u+1}$ and $E_{v,v+1}$ have no edges in common.*

Proof. From Fig. 2 we see that the edges in $E_{u,u+1}$ and $E_{v,v+1}$ are all different if $v \notin \{u, u+2m, u+2m+1\}$, where addition is modulo $4m+1$. The construction of the set S is divided into the three cases of $m \equiv 0 \pmod 3$, $m \equiv 1 \pmod 3$, and $m \equiv 2 \pmod 3$.

If $m \equiv 0 \pmod 3$:

$$S = \{0\ 1,\ 3\ 4,\ 6\ 7, \ldots, 2m-3\ 2m-2,\ 2m+2\ 2m+3,$$
$$2m+5\ 2m+6, \ldots, 4m-1\ 4m\}.$$

There are $4m/3$ edges in S.

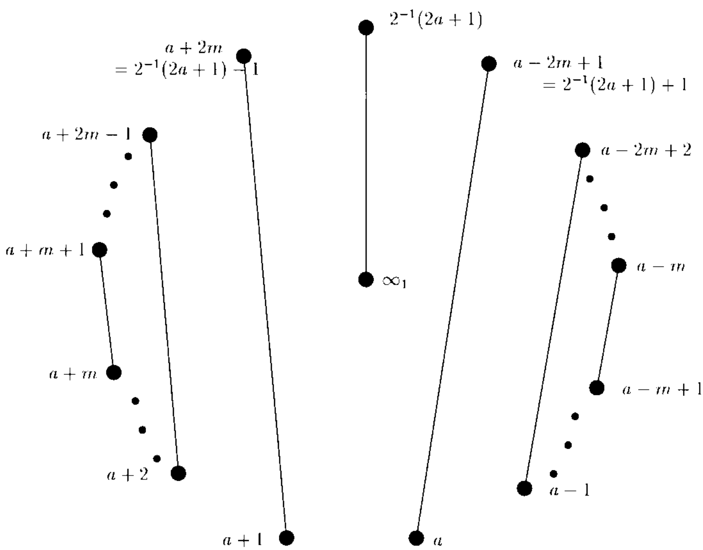

FIG. 2. The edges in $E_{a,a+1}$.

If $m \equiv 1 \pmod 3$ and $m > 1$:

$$S = \{0\,1, 3\,4, 6\,7, ..., 2m-2\;2m-1, 2m+2\;2m+3,$$
$$2m+5\;2m+6, ..., 4m-3\;4m-2\}.$$

There are $(4m-1)/3$ edges in S.

If $m \equiv 2 \pmod 3$:

$$S = \{0\,1, 3\,4, 6\,7, ..., 4m-2\;4m-1\}.$$

There are $(4m+1)/3$ edges in S. ∎

LEMMA 19. *For $m > 1$, there are at least $\lfloor (4m+1)/3 \rfloor$ pairwise compatible Hamilton decompositions of K_{4m+3}.*

Proof. Consider the set S constructed in Lemma 18. Choose an edge $a\,a+1$ in S. Use Lemma 14 to construct a set of trails based on F_a and F_{a+1}. If we add the 2-paths $\{c \infty d : c\,d \in E_{a,\,a+1}\}$ to these trails, we obtain a Hamilton decomposition of K_{4m+3}. Do this for each edge in S to obtain $\lfloor (4m+1)/3 \rfloor$ Hamilton decompositions of K_{4m+3}. The 2-paths centered at any vertex in $V(K_{4m+2})$ will all be different because the edges in S are disjoint; the 2-paths centered at ∞ will all be different because, by Lemma 18, for any two edges $u\,u+1$, $v\,v+1 \in S$, the sets $E_{u,\,u+1}$ and $E_{v,\,v+1}$ have no edge in common. ∎

LEMMA 20. *The graph K_7 has at least two compatible Hamilton decompositions.*

Proof. Let $V(K_7) = \{0, 1, 2, 3, 4, 5, 6\}$. The following are two compatible Hamilton decompositions of K_7:

$$(0, 1, 2, 3, 4, 5, 6), (0, 2, 4, 6, 1, 3, 5), (0, 3, 6, 2, 5, 1, 4),$$
$$(0, 2, 5, 6, 4, 3, 1), (0, 3, 2, 6, 1, 4, 5), (0, 4, 2, 1, 5, 3, 6). \quad ∎$$

Theorem 16 now follows immediately from Lemmas 19 and 20. ∎

This completes the case of k odd. We now want to construct pairwise compatible Hamilton decompositions of K_{2k+1} when k is even. From now on, we assume $k = 2m$.

THEOREM 21. *Let $m > 1$. There are at least* $\max(\lceil 4m/3 \rceil - (2m - 1 - \phi(4m-1)/2), 3)$ *pairwise compatible Hamilton decompositions of K_{4m+1}. There are no two compatible Hamilton decompositions of K_5.*

The proof of this theorem requires the following two lemmas.

LEMMA 22. *Given $a, b \in V(K_{4m}) \backslash \{\infty_1\}$, the length of any edge in $E_{a,b}$ that does not contain ∞_1 is the constant $\ell(2^{-1}(a-b)\,0)$.*

Proof. By Lemma 14, if we start a trail on the edge $\infty\, w_i$, $2 \leqslant i \leqslant 2m$, it will finish on the edge $w_{i+2m}\infty$, where addition on the subscripts is modulo $4m$, with residue classes $1, 2, ..., 4m$. By definition of F_a and F_b, if i is even, $w_{i+2m} \equiv 2m(a-b) + w_i \equiv 2^{-1}(a-b) + w_i \pmod{4m-1}$. If i is odd, then $w_{i+2m} \equiv 2m(b-a) + w_i \equiv 2^{-1}(b-a) + w_i \pmod{4m-1}$. In either case, $\ell(w_i\, w_{i+2m}) \equiv \ell(2^{-1}a - b)\,0)$. ∎

LEMMA 23. *If $m > 1$, then there exists a set S of $\lceil 4m/3 \rceil$ disjoint edges in K_{4m} such that:*

1. *No two of the edges are parallel,*
2. *No two of the edges have the same length, and*
3. *None of the edges contains the vertex ∞_1.*

Moreover, we can always find a subset S^* of S with at least three edges that have lengths relatively prime to $4m-1$. (If $m=1$ there is only one such edge.)

Proof. The proof is divided into the three cases of $m \equiv 0 \pmod 3$, $m \equiv 1 \pmod 3$, and $m \equiv 2 \pmod 3$.

If $m \equiv 0 \pmod 3$:

$$S = \left\{0\ 2m-1,\ 1\ 2m-3,\ 2\ 2m-5,\ ...,\ \frac{2m}{3}-1\ \frac{2m}{3}+1\right\}$$
$$\cup \left\{4m-2\ 2m+1,\ 4m-3\ 2m+3,\ 4m-4\ 2m+5,\ ...,\ \frac{10m}{3}\ \frac{10m}{3}-3\right\}$$
$$\cup \{2m-2\ 2m+2\}.$$

The set S has $4m/3$ edges. Let $S^* = \{0\ 2m-1,\ 2m/3-1\ 2m/3+1,\ 2m-2\ 2m+2\}$.

If $m \equiv 1 \pmod 3$:

$$S = \left\{0\ 2m-1,\ 1\ 2m-3,\ 2\ 2m-5,\ ...,\ \frac{2m-2}{3}\ \frac{2m+1}{3}\right\}$$
$$\cup \left\{4m-2\ 2m+1,\ 4m-3\ 2m+3,\ 4m-4\ 2m+5,\ ...,\ \frac{10m-1}{3}\ \frac{10m-7}{3}\right\}$$
$$\cup \{2m-4\ 2m+2\}.$$

In this case, S has $(4m+2)/3$ edges if $m>1$. (It has only one edge if $m=1$.) Let

$$S^*=\left\{0\ 2m-1,\ \frac{2m-2}{3}\ \frac{2m+1}{3},\ \frac{10m-1}{3}\ \frac{10m-7}{3}\right\}$$

when $m>1$.

If $m \equiv 2 \pmod 3$:

$$S=\left\{0\ 2m-1,\ 1\ 2m-3,\ 2\ 2m-5,\ \ldots,\ \frac{2m-4}{3}\ \frac{2m+5}{3}\right\}$$

$$\cup\left\{4m-2\ 2m+1,\ 4m-3\ 2m+3,\ 4m-4\ 2m+5,\ \ldots,\ \frac{10m-2}{3}\ \frac{10m-5}{3}\right\}$$

$$\cup\{2m-2\ 2m\}.$$

In this case S has $(4m+1)/3$ edges. Let

$$S^*=\left\{0\ 2m-1,\ \frac{10m-2}{3}\ \frac{10m-5}{3},\ 2m-2\ 2m\right\}. \quad \blacksquare$$

Proof of Theorem 21. Assume $m>1$. By Lemma 23 we can find a set S of $\lceil 4m/3 \rceil$ disjoint edges in K_{4m} so that no two of the edges are parallel, no two of the edges have the same length, and none of the edges contains ∞_1. There are at least $\lceil 4m/3 \rceil-(2m-1-\phi(4m-1)/2)$ disjoint edges $a\,b \in S$ such that $(a-b,\,4m-1)=1$. If $\lceil 4m/3 \rceil-(2m-1-\phi(4m-1)/2) \geqslant 3$, choose S' to be this subset of S. If $\lceil 4m/3 \rceil-(2m-1-\phi(4m-1)/2)<3$, choose S' to be the set S^* defined in Lemma 23, so that $|S'|$ is always at least 3. Consider an edge $a\,b \in S'$. Since $\infty_1 \notin \{a, b\}$, both F_a and F_b are defined and, by Lemma 7, we know that $F_a \cup F_b$ is a Hamilton cycle. Use Lemma 6 to construct a set of trails of K_{4m+1} based on $F_a \cup F_b$, such that the trail that starts on the edge $\infty\ \infty_1$ ends on the edge $(2^{-1}(a+b))\ \infty$. By Lemma 22 the length of each the edges, $\{w_i\, w_{i+2m}\colon 2 \leqslant i \leqslant 2m\}$, determined by the first and last vertices of each of the other trails is a constant, $\ell(2^{-1}(a-b)\ 0)$, dependent on the length of the edge $a\,b$. We can extend these trails to Hamilton cycles of K_{4m+1} by adding the 2-paths $\{c\ \infty\ d : c\, d \in E_{a,b}\}$ to close them off. These Hamilton cycles together comprise a Hamilton decomposition of K_{4m+1}. Doing this for each such edge $a\,b \in S'$ gives $\max(\lceil 4m/3 \rceil-(2m-1-\phi(4m-1)/2), 3)$ Hamilton decompositions of K_{4m+1}. Since the edges in S' are disjoint, the end vertices of 2-paths centred at any vertex $v \in V(K_{4m})$ come from different 1-factors in each of the Hamilton path decompositions. Since no two edges in S have the same length, all the 2-paths centered at ∞ that do not contain ∞_1 will be distinct. And since none of the edges in S are parallel, we know by Lemma 15 that all the 2-paths centered at ∞ that do contain ∞_1 will be distinct.

It is not hard to show that there cannot be two compatible Hamilton decompositions of K_5. ∎

Given m, we can possibly do better than Theorem 21 by actually counting the number of edges in the set S that have lengths relatively prime to $4m-1$. Also, given m, we could deliberately construct a set $S^\dagger$, as in the following corollary, so as to improve the number of pairwise compatible Hamilton decompositions.

COROLLARY 24. *Let $S^\dagger$ be any set of disjoint edges in K_{4m} such that ∞_1 is not in any of the edges, no two of the edges are parallel, no two of the edges have the same length, and such that $(a-b, 4m-1)=1$ for all edges $ab \in S^\dagger$. There are at least $|S^\dagger|$ pairwise compatible Hamilton decompositions of K_{4m+1}.*

More specifically, if $4m-1$ is prime, then the union of any two of the 1-factors of K_{4m} is a Hamilton cycle.

COROLLARY 25. *Suppose $4m-1$ is prime. Then there at least $\lceil 4m/3 \rceil$ pairwise compatible Hamilton decompositions of K_{4m+1}.*

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